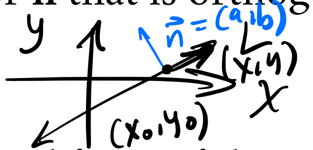
$n! = n(n-1)\cdots 3\cdot 2\cdot 1$ $\quad 0! = 1$

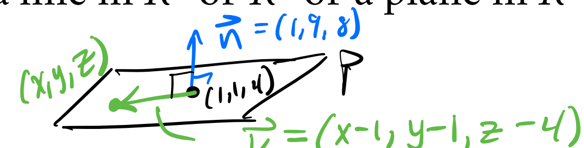
3.3 – Orthogonality $\quad 4! = 24$

Definition: Two nonzero vectors **u** and **v** in R^n are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u}\cdot\mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to every vector in R^n.

$(a,b)\cdot(x-x_0, y-y_0) = 0 \Rightarrow a(x-x_0)+b(y-y_0) = 0$

Definition: A vector **n** that is orthogonal to a line in R^2 or R^3 or a plane in R^3 is called a **normal**.





#4 Find a point-normal form of the equation of the plane passing through P and having **n** as a normal.
$P(1, 1, 4)$, $\mathbf{n} = (1, 9, 8)$

$$\vec{n}\cdot\vec{v} = 0 \Rightarrow (1,9,8)\cdot(x-1, y-1, z-4) = 0$$

says Reece $\quad 1(x-1) + 9(y-1) + 8(z-4) = 0$

$$x + 9y + 8z - 42 = 0$$

Theorem 3.3.1
a) If a and b are constants that are not both zero, then an equation of the form $ax + by + c = 0$ represents a line in R^2 with normal $\mathbf{n} = (a, b)$.
b) If a, b, and c are constants that are not all zero, then an equation of the form $ax + by + cz + d = 0$ represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

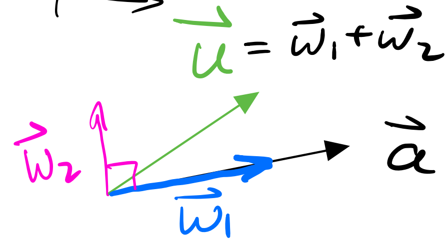
Theorem 3.4.3 (not a typo)
If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.

$$(a_{11}, a_{12}, \ldots, a_{1n}) \cdot (x_1, x_2, \ldots, x_n) = 0$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\quad &\ \ \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Theorem 3.3.2 Projection Theorem $\longrightarrow R^2:$



If $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^n, and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

$\vec{w}_1$ parallel to $\vec{a}$ means

$$\vec{w}_1 = k\vec{a} \Rightarrow \vec{u} \cdot \vec{a} = (\vec{w}_1 + \vec{w}_2) \cdot \vec{a}$$

$$= k\vec{a} \cdot \vec{a} = k\|\vec{a}\|^2$$

So $k = \dfrac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}$ Then $\vec{w}_1 = \dfrac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$

$$\vec{w}_1 + \vec{w}_2 = \vec{u} \Rightarrow \vec{w}_2 = \vec{u} - \vec{w}_1$$

$$\vec{w}_2 = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

Side note: $\vec{u} \cdot \vec{a} = \|\vec{u}\| \|\vec{a}\| \cos\theta$

$$\frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \|\vec{u}\| \cos\theta \frac{\vec{a}}{\|\vec{a}\|}$$

Definition: When $\mathbf{u}$, a vector in R^n, is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of a vector $\mathbf{a}$ in R^n, $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \dfrac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **orthogonal projection of u on a** or the **vector component of u along a**. The vector $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \dfrac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **vector component of u orthogonal to a**.

$\vec{w}_1$

#20 Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a.** $\vec{w}_2$

$\mathbf{u} = (5, 0, -3, 7)$, $\mathbf{a} = (2, 1, -1, -1)$

$$\vec{w}_1 = \text{proj}_{\vec{a}}\, \vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}, \quad \vec{w}_2 = \vec{u} - \vec{w}_1$$

$$\vec{w}_1 = \frac{6}{7}(2,1,-1,-1) = \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7}\right)$$

$$\vec{w}_2 = (5,0,-3,7) - \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7}\right)$$

$$\vec{w}_2 = \left(\frac{23}{7}, -\frac{6}{7}, -\frac{15}{7}, \frac{55}{7}\right)$$

$\vec{w}_1 \cdot \vec{w}_2 = ?$ $\quad \vec{w}_2 \cdot \vec{a} = \frac{46}{7} - \frac{6}{7} + \frac{15}{7} - \frac{55}{7} = 0$

$$\vec{w}_2 \cdot \vec{a} = 0 \Rightarrow \vec{w}_2 \perp \vec{a}$$

Proj onto x-axis: $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ Rotation $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

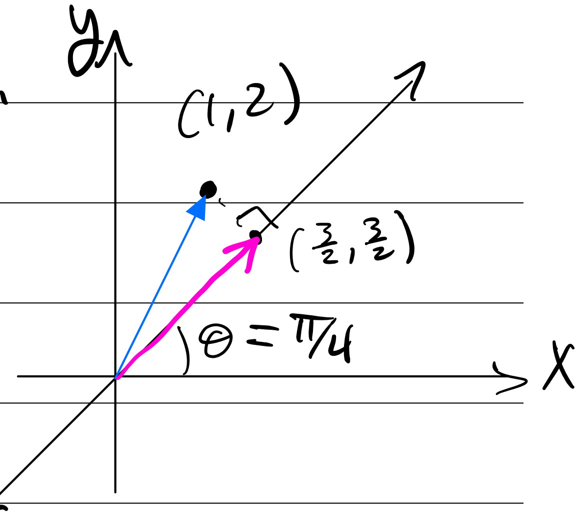
Refl. about y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

#38 Find the standard matrix for the orthogonal projection of R^2 onto the stated line, and then use that matrix to find the orthogonal projection of the given point onto that line.

The orthogonal projection of $(1, 2)$ onto the line that makes an angle of $\pi/4$ ($= 45^\circ$) with the positive x-axis.

Standard matrix for transformation

$$T_A = \left[T(\vec{e}_1) \,\middle|\, T(\vec{e}_2) \,\middle|\, \cdots \,\middle|\, T(\vec{e}_n) \right]$$



use $\vec{a} = (\cos\theta, \sin\theta)$

$$\text{proj}_{\vec{a}}\, \vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{a}}{\|\vec{a}\|^2}\, \vec{a} =$$

$$= \cos\theta(\cos\theta, \sin\theta) = (\cos^2\theta,\ \cos\theta\sin\theta)$$

$$\text{proj}_{\vec{a}}\, \vec{e}_2 = \sin\theta(\cos\theta, \sin\theta) = (\sin\theta\cos\theta,\ \sin^2\theta)$$

$$T_A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \qquad \theta = \pi/4$$

$$T_A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad T_A\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

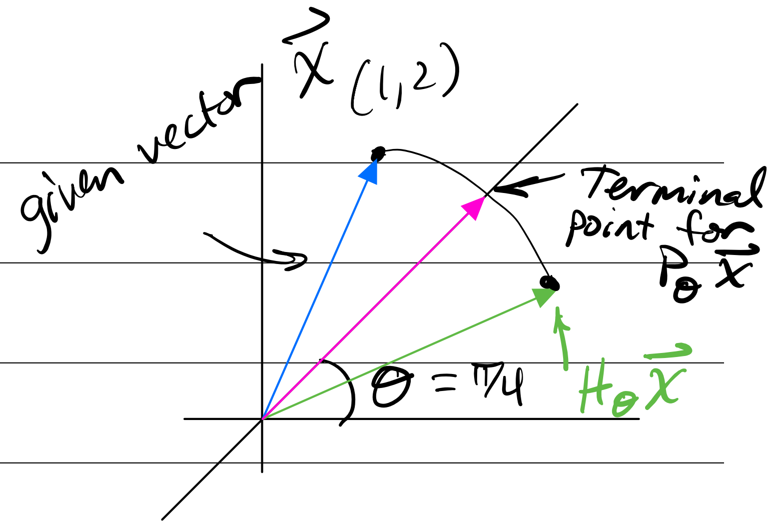
$$= \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

In general, the matrix A is denoted by

$$P_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$

#36 Find the standard matrix for the reflection of R^2 about the stated line, and then use that matrix to find the reflection of the given point about that line.

The reflection of $(1,2)$ about the line that makes an angle of $\pi/4$ $(=45^\circ)$ with the positive x-axis.



The distance between $P_\theta \vec{x}$ and $\vec{x}$ is half the distance between $H_\theta \vec{x}$ and $\vec{x}$. Thus

$$P\vec{x} - \vec{x} = \frac{1}{2}(H_\theta \vec{x} - \vec{x})$$

$$H_\theta \vec{x} = 2P\vec{x} - \vec{x} = (2P_\theta - I)\vec{x}$$

$$H_\theta = 2P_\theta - I = \begin{bmatrix} 2\cos^2\theta - 1 & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \Rightarrow H_{\pi/4} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$H_\theta \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Theorem 3.3.3 Theorem of Pythagoras in R^n

If **u** and **v** are orthogonal vectors in R^n with the Euclidean inner product, then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

pf: $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$

But $\vec{u}\cdot\vec{v} = 0$, so we get $\|\vec{u}\|^2 + \|\vec{v}\|^2$ ✓